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CONNECTION DIGRAPHS AND SECOND-ORDER LINE DIGRAPHS

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The primary result of this paper gives a set of necessary and sufficient conditions for a digraph to be the second-order line digraph of some digraph. A directed analogue of the concept of an intersection graph, defined for collections of ordered pairs of sets and called the connection digraph, is used to achieve this result.

Several sets of necessary and sufficient conditions for a digraph to be a line digraph were found during the 60's (see Theorem 1 below); those which are most notable were discovered by Harary and Norman [2], Heuchenne [5], and Richards [6]. Heuchenne's condition is a local structural condition, and it was reported [3] that a natural extension of his result holds for nth-order iterated line digraphs. Although those extended conditions are necessary, they are not sufficient, and the primary objective of this paper is to present a set of necessary and sufficient conditions for a digraph to be a second-order line digraph.

The paper has three sections: the first presents some background material and introduces concepts and notation, including the connection digraph, needed for our result; the second consists of our main theorem and its proof; and the third contains related results for some special classes of digraphs.

1. Preliminaries

The *line digraph* $L(D)$ of a digraph D has vertices corresponding to the arcs of D, with an arc from one vertex to another if the corresponding arcs are such that the head of the first is the tail of the second. An example of a digraph and its line digraph is shown in Fig. 1.

Generally, and unless it is stated otherwise, we shall allow digraphs to have loops and multiple arcs. This convention allows certain results to be stated more easily than if the digraphs are not allowed to contain such arcs. We observe, however, that line digraphs cannot have multiple arcs, and they have loops if and only if the original digraphs have.

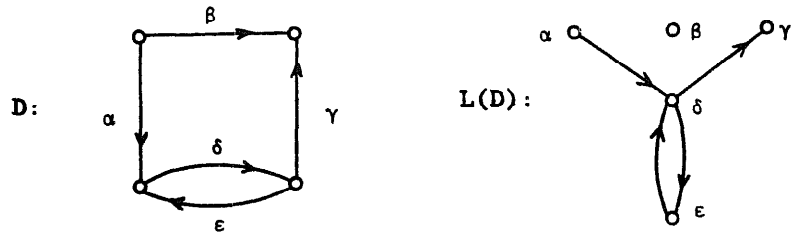


Fig. 1.

The following theorem gives the characterizations of line digraphs referred to earlier. In the theorem, (2) is due to Harary and Norman [2], (3) to Heuchenne [5], and (4) to Richards [6]. The following notation is used: if S and T are sets of vertices (not necessarily disjoint), then $\vec{K}(S, T)$ denotes the digraph with vertex-set $S \cup T$ and arc-set $S \times T$. Also, a *general partition* of a set is a decomposition into pairwise disjoint subsets, some of which may be empty. For a unified proof of this theorem, see Hemminger and Beineke [4].

Theorem 1. *The following statements are equivalent for a digraph D having no multiple arcs.*

(1) *D is a line digraph.*

(2) *There exist two general partitions $\{S_\lambda\}_{\lambda=1}^n$ and $\{T_\lambda\}_{\lambda=1}^n$ of the vertex-set of D so that the arc-sets of the collection $\{\vec{K}(S_\lambda, T_\lambda)\}_{\lambda=1}^n$ form a general partition of the arc-set of D.*

(3) *If, for vertices u, v, w, and x (not necessarily distinct), D contains the arcs $u \to w$, $v \to w$, and $v \to x$, then D must contain the arc $u \to x$.*

(4) *Any two rows (or any two columns) of the adjacency matrix of D are either identical or orthogonal.*

Because (2) is directly related to some of the material to follow, we consider an example in some detail. Fig. 2 shows a digraph D, an arc-decomposition into three subgraphs of the type $\vec{K}(S, T)$ and a digraph E of which D is the line digraph.

The digraph E is obtainable from the arc-decomposition by a construction which is a directed analogue of the formation of (undirected) intersection graphs We define the connection digraph of an arbitrary collection of ordered pairs of sets as follows: In the *connection digraph of a collection* $\{(S_\lambda, T_\lambda)\}_{\lambda=1}^m$, there is a vertex v_λ for each pair of sets (S_λ, T_λ), and there is an arc from v_i to v_j for each element in $T_i \cap S_j$.

We observe that every digraph is the connection digraph of some collection: For each vertex v in a digraph D, take the ordered pair $(A^-(v), A^+(v))$, where $A^-(v)$ is the set of in-coming arcs at v and $A^+(v)$ is the set of out-going arcs. It can be readily verified that the connection digraph of this family is isomorphic to D.

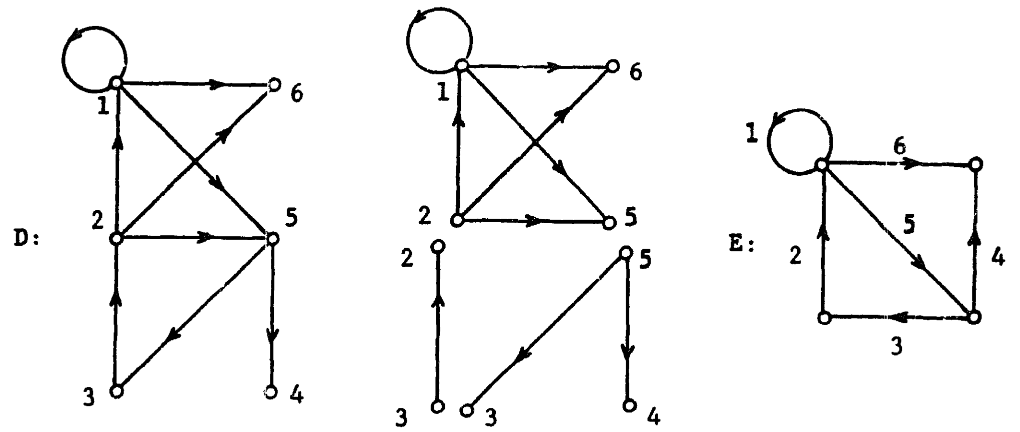


Fig. 2.

Theorem 2. *Every digraph is a connection digraph.*

Returning to our example, we consider the pairs of sets

$$v_1: \ (\{1, 2\}, \{1, 5, 6\}), \qquad v_2: \ (\{3\}, \{2\}),$$
$$v_3: \ (\{5\}, \{3, 4\}), \qquad v_4: \ (\{4, 6\}, \emptyset).$$

This family constitutes the two partitions of the vertex-set of D, and E is the connection digraph of the family. This relationship between line digraphs and connection digraphs always holds, and in fact the sufficiency of condition (2) of the theorem may be established by constructing connection digraphs. This was done, for example, in the proof of Theorem 1 given in [4] (where it appears as Theorem 8.4). Thus, the following result is implicitly contained in that theorem and proof. For convenience, we define a collection of pairs $\{(S_\lambda, T_\lambda)\}_{\lambda=1}^m$ of subsets of the vertex-set of a digraph D to be a *double partition* if both $\{S_\lambda\}_{\lambda=1}^m$ and $\{T_\lambda\}_{\lambda=1}^m$ are general partitions of the vertex set and if $\{K(S_\lambda, T_\lambda)\}_{\lambda=1}^m$ constitutes a general partition of the arc-set.

Theorem 3. *Let D and E be a digraph. Then D is the line digraph of E iff E is isomorphic to the connection digraph of some double partition of D.*

In our example, there is a second digraph of which D is the line digraph, and it is the connection digraph of this double partition:

$$(\{1, 2\}, \{1, 5, 6\}), \quad (\{3\}, \{2\}), \quad (\{5\}, \{3, 4\}), \quad (\{4\}, \emptyset), \quad (\{6\}, \emptyset).$$

In general, the arcs of a line digraph have a unique (proper) partition into product sets $S_\nu \times T_\nu$, so that the only flexibility in constructing all the pairs (S_ν, T_ν) of a double partition of the vertex-set occurs in the placement of vertices with in-valency or out-valency 0, in which cases the other set of a pair is empty. The

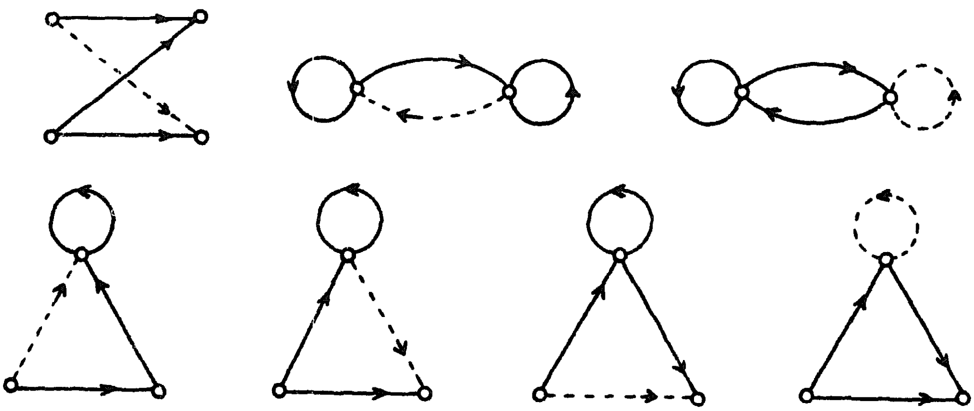


Fig. 3.

construction of these sets will form a crucial step in the proof of our main result later in the paper.

We now turn our attention to condition (3) of Theorem 1, which provides a local criterion for a digraph to be a line digraph. This condition is shown structurally in Fig. 3: if a line digraph contains a subgraph with the three solid lines of any of the seven digraphs, then it must also contain the dotted line.

Hemminger [3] called this property the *(first) Heuchenne condition*, and he observed that a generalization of this property must be satisfied by any nth-order line digraph. Formally, we make the following definitions. The *nth-order line digraph* $L^n(D)$ of a digraph D is defined inductively with L denoting the first-order operation and $L^n(D)=L(L^{n-1}(D))$. A digraph satisfies the *nth Heuchenne condition* if it has this property: For any vertices, u, v, w, and x (not necessarily distinct) for which there exist n-diwalks[1] from u to w, from v to w, and from v to x, there must also exist an n-diwalk from u to x. (See Fig. 4.) While an nth-order line digraph must satisfy the kth Heuchenne condition for all $k \leq n$ (and for each $k \leq n$ can have no two k-diwalks from one vertex to another) these conditions are not sufficient for a digraph to be an nth-order line digraph. In particular, for $n=2$, the two digraphs in Fig. 5 satisfy the first two Heuchenne conditions (as well as having no multiple arcs and no pairs of arc-disjoint 2-diwalks from one vertex to another), but neither is a second-order line digraph.

Fig. 6 shows the minimal second-order line digraphs which contain the digraphs of Fig. 5. What was needed was different in the two cases: the first required a vertex of in-valency 1 and out-valency 0 (appropriately attached), while the second needed just an isolated vertex. Theorem 4 provides for such vertices and establishes the fact that these are essentially the only additions needed for a characterization. A vertex which has positive out-valency and zero in-valency is called a *source*, while a vertex with positive in-valency and zero out-valency is called a *sink*. A vertex with both in- and out-valency 0 is called

[1] Please note that the arcs of an n-diwalk are also not necessarily distinct.

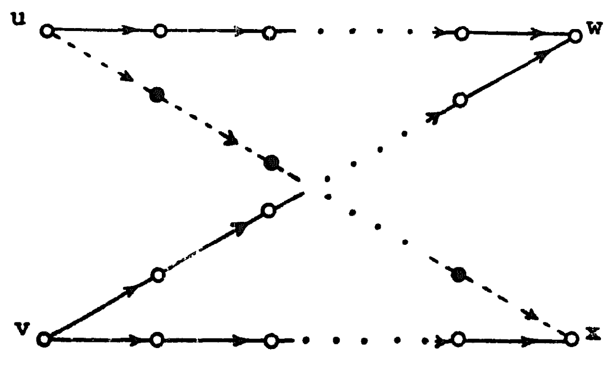


Fig. 4.

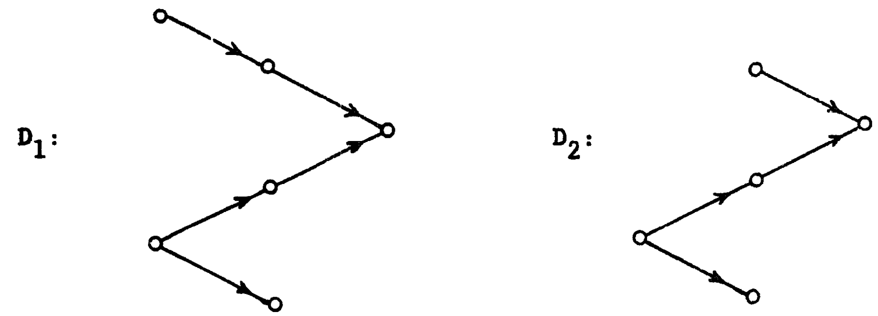


Fig. 5.

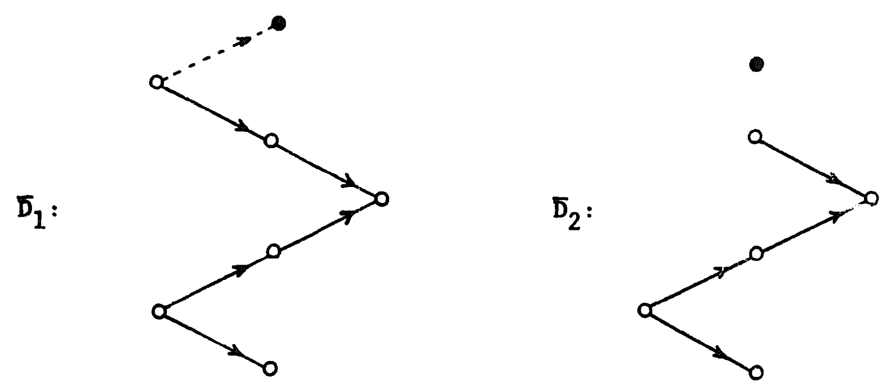


Fig. 6.

isolated. Condition (3) in the theorem provides for sources and sinks, while (4) provides for isolated vertices.

Before getting to our theorem, we introduce still further concepts leading to a particular type of subgraph of a digraph. For two 2-diwalks $P: u_0 \to u_1 \to u_2$ and $Q: v_0 \to v_1 \to v_2$, we say that P and Q are *hooked* if $u_i = v_i$ for at least one value of i. Further, we define the relation $P \sim Q$ if there is a sequence $P_0 = P$, $P_1, \ldots, P_k = Q$ of 2-diwalks such that P_{i-1} and P_i are hooked, for all i. It is clear that $\sim$ is an equivalence relation on 2-diwalks. The union of the 2-diwalks in an $\sim$-equivalence class is called a *tri-level subgraph*, and the first vertices of the

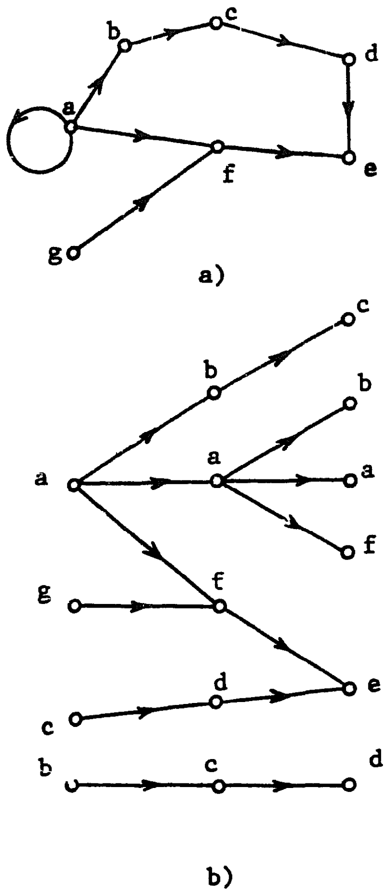


Fig. 7.

2-diwalks constitute the *first-level*, the second vertices the *middle-level*, and the third vertices the *last-level* (note that one vertex may have several levels).

An example of a tri-level digraph is shown in Fig. 7. Consider the digraph exhibited in Fig. 7(a). Fig. 7(b) puts in evidence the three levels of the vertices of this digraph (the digraph from Fig. 7(a) can then be obtained from the one in Fig. 7(b) by identifying the vertices labeled with the same letters).

We note that this digraph does not satisfy the first two Heuchenne conditions and therefore it can not be a second-order line digraph. However, the digraphs in Fig. 8 show that it can be completed so that those conditions are satisfied.

2. The main result

Theorem 4. *A line digraph is a second-order line digraph iff the following conditions are satisfied.*

(1) *For all vertices a and b, there is at most one 2-diwalk from a to b.*

(2) *For all vertices a, b, c, and d, if there are 2-diwalks from a to c, from b to c, and from b to d, then there is a 2-diwalk from a to d (the second Heuchenne condition).*

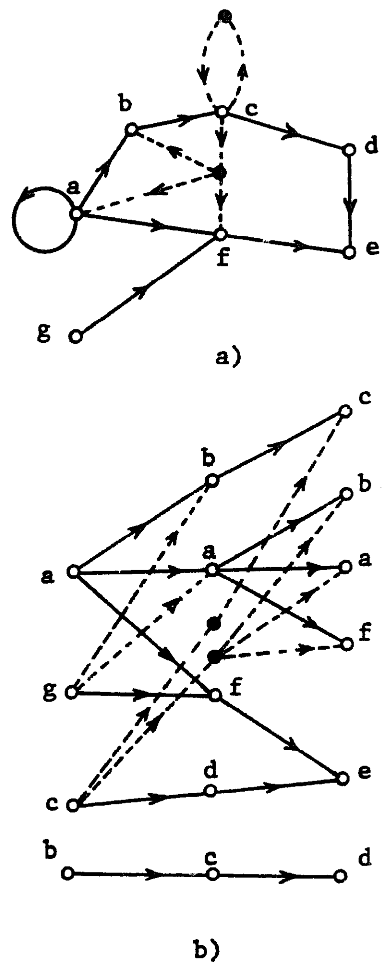


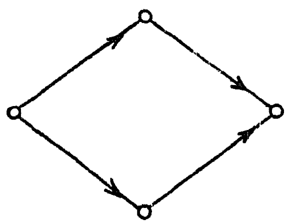
Fig. 8.

(3) *In any tri-level subgraph, all first-level vertices have the same number of sink neighbors, and (dually) all last-level vertices have the same number of source neighbors.*

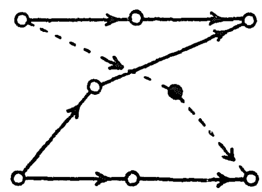
(4) *For each tri-level subgraph, there corresponds at least pq isolated vertices, where p is the number of sink neighbors of each first-level vertex and q is the number of source neighbors of each last-level vertex and different tri-level subgraphs have disjoint sets of associated isolated vertices.*

Before proving the theorem, we refer to Fig. 9, which gives some pictorial representations illustrating the four conditions. We also remind the reader that in the statements of the conditions not all of the vertices need be distinct.

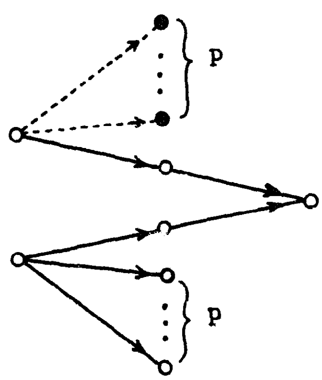
Proof. We first establish the necessity of the four conditions, and to this end, we assume that D is the second-order line digraph of F and that E denotes the first-order line digraph of F. It follows from Theorem 3 that there is a double partition $\{(S_\lambda, T_\lambda)\}_{\lambda=1}^{m}$ of the vertex-set of D (which thus partitions the arc-set)



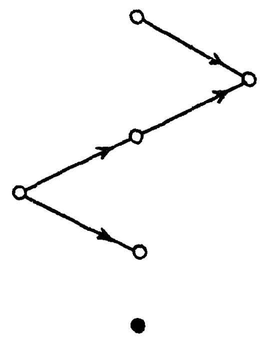
(1) a forbidden configuration



(2) a necessary 2-dipath



(3) equinumerous sink neighbors



(4) a necessary isolated vertex

Fig. 9.

and for which E is the connection digraph. We let v_λ denote the vertex of E corresponding to (S_λ, T_λ).

We shall use the following lemma, whose simple proof is omitted:

Lemma. *Given $x \in S_i$ and $z \in T_j$, there is a vertex y such that $x \to y \to z$ is in D iff $v_i \to v_j \in E$. Moreover, the number of such vertices in D equals the number of such edges in E.*

To show (1), we suppose that D has two distinct 2-diwalks from b to c: $b \to v \to c$ and $b \to w \to c$. Then there must exist integers i and j such that $b \in S_i$ and $c \in T_j$. It follows that v and w are both in T_i and S_j and hence there are two arcs from v_i to v_j in E. However, this is impossible since E is a line digraph, and therefore condition (1) must hold.

To establish (2), we assume that D contains the 2-diwalks $a \to u \to c$, $b \to v \to c$, and $b \to w \to d$. Let $a \in S_h$, $b \in S_i$, $c \in T_j$, and $d \in T_k$. From the lemma it follows that E contains the arcs $v_h \to v_j$, $v_i \to v_j$, and $v_i \to v_k$. Since E is a line digraph, by the first Heuchenne condition, E must contain the arc $v_h \to v_k$ which, by the lemma, implies that there is a 2-diwalk from a to d in D.

For (3), it is sufficient, by duality and the transitivity of $\sim$, to show that if there are hooked 2-diwalks $a \to u \to c$ and $b \to v \to d$ in D, with $a \neq b$, then a and b

have the same number of sink neighbors. For, then any two $\sim$-related 2-diwalks must have that property, and furthermore the dual statement holds by duality.

We observe that if $u = v$, then a and b must have the same set and hence number of sink neighbors by virtue of the first Heuchenne condition. Hence we assume $u \neq v$ and so $c = d$. We let w be a sink neighbor of b and let $a \in S_h$, $b \in S_i$, $c \in T_j$, and $w \in S_k$. Thus $T_k = \emptyset$. We will show that there is exactly one sink neighbor of a in the set S_k. Since $w \in T_i \cap S_k$, E contains the arc $v_i \to v_k$ and, by the lemma, E contains the arcs $v_h \to v_j$ and $v_i \to v_j$. By the first Heuchenne condition E contains the arc $v_h \to v_k$. Therefore, D must have a vertex x in both T_h and S_k. Since all sink neighbors of a must be in T_h, it follows that S_k cannot contain another one (since if it did, E would have multiple arcs). For the same reason, S_k contains exactly one sink neighbor of b. Similarly every such S_k contains the same number of sink neighbors of a and b. This serves to establish (3).

In proving (4), we first show that if $b \to v \to c$ is a 2-diwalk, if w is a sink neighbor of b and if u is a source neighbor of c, then there is an isolated vertex x in the set S_k containing w and the set T_h containing u. Let $b \in S_i$ and $c \in T_j$. We have v, $w \in T_i$, u, $v \in S_j$, $T_k = S_h = \emptyset$. As before, from the lemma and the first Heuchenne condition, follows that E must contain the arc $v_h \to v_k$, and hence D must have a vertex x in $T_h \cap S_k$. Furthermore, since both S_h and T_k are empty, x must be an isolated vertex.

Since E cannot have multiple arcs, it follows, as in the latter part of the proof of case (3), that every such S_k contains exactly one sink neighbor of b. Similarly every such T_h contains exactly one source neighbor of a. Hence if b has p sink neighbors and c and q source neighbors then there must be at least pq associated isolated vertices.

All that remains is to show that an isolated vertex x for the tri-level subgraph of $b \to v \to c$ cannot serve for another tri-level subgraph. To this end, we assume that there is a walk $w' \leftarrow b' \to v' \to c' \leftarrow u'$ with w' a source in S_k (which contains w and x) and u' a sink in T_h (which contains u and x). Let i' and j' be such that $b' \in S_{i'}$, and v', $w' \in T_{i'}$, and u', $v' \in S_{j'}$ and $c' \in T_{j'}$. It follows as before that E must contain arcs $v_{i'} \to v_{j'}$, $v_h \to v_{j'}$, and $v_{i'} \to v_k$. Because of the $b \to v \to c$ tri-level subgraph, we also have $v_i \to v_j$, $v_h \to v_j$, and $v_i \to v_k$ in D, so from the first Heuchenne condition it follows that E contains the arcs $v \to v_{j'}$ and $v_{i'} \to v_j$. Hence, D must contain vertices y and z with $y \in T_i \cap S_{j'}$ and $z \in T_{i'} \cap S_j$. From the lemma we have $b \to y \to c'$ and $b' \to z \to c$. It follows that $b \to v \to c \sim b \to y \to c' \sim b' \to v' \to c'$. Therefore, an isolated vertex can serve only one tri-level subgraph, a fact which completes the proof of the necessity.

To prove the sufficiency, we assume that D is a line digraph which satisfies conditions (1)–(4) and proceed to construct from D a connection digraph E in such a way that it too will be a line digraph. Since D is a line digraph, it has double partitions, and therefore we must specify one for which the connection digraph has no multiple arcs and satisfies the first Heuchenne condition.

For a given double partition $\{(S_\lambda, T_\lambda)\}_{\lambda=1}^m$ of D, it is easily seen that if $v \in S_\lambda$ has positive out-valency, then S_λ is determined (as the set of vertices with arcs to precisely the same vertices as v), and similarly that if $v \in T_\lambda$ has positive in-valency, then T_λ is determined. Therefore, in constructing our double partition, the only flexibility is in partitioning the sinks into S_λ sets (the corresponding $T_\lambda = \emptyset$) and the sources into T_λ sets (the corresponding $S_\lambda = \emptyset$).

We begin with the sinks. First consider sinks which have arcs to them from a first-level vertex of a given tri-level subgraph. We observe that each of these sinks is so connected with only one tri-level subgraph. For, since D satisfies the first Heuchenne condition, all neighbors of such a sink must be first-level vertices in the same tri-level subgraph. Thus, we consider one tri-level subgraph and the sink neighbors of its first-level vertices. The sets of sink neighbors of two first-level vertices must be either identical or disjoint, again because of the first Heuchenne condition. Therefore, each sink neighbor of this tri-level subgraph can be assigned a number from 1 to p (where p is the common number of sink neighbors of the first-level vertices) in such a way that the sink neighbors of each first-level vertex have different numbers. We now take all sink neighbors of first-level vertices in this tri-level subgraph with the same assigned number to belong to one set S_λ. Thus sink neighbors of different tri-level subgraphs will always be in different S_λ.

The assignment of sources (which have arcs to a last-level vertex in some tri-level subgraph) to sets T_λ is made similarly.

If a set S_i containing sinks and a set T_j containing sources as just defined arise from the same tri-level subgraph, we extend S_i and T_j by assigning them a common isolated vertex (by (4), there are enough isolated vertices to do this). To each remaining sink z (source w) we assign $(S_\lambda, T_\lambda) = (\{z\}, \emptyset)$ (respectively $(\emptyset, \{w\})$).

Having thus defined our double partition $\{(S_\lambda, T_\lambda)\}_{\lambda=1}^m$ (which also partitions the arc-set), we must show that its connection digraph E is itself a line digraph.

First we prove that E has no multiple arcs. Our construction of the partitions prevents all the following, for all S_i and T_j:

two sink neighbors of one vertex in the same S_i,

two source neighbors of one vertex in the same T_j,

two isolated vertices in the same S_j and T_j.

Therefore, multiple arcs in E would of necessity be the result of multiple 2-diwalks in D, and this is prohibited by condition (1). Hence, E can have no multiple arcs.

Next we assume that E contains vertices v_1, v_2, v_3, and v_4 such that the arcs $v_1 \to v_3$, $v_2 \to v_3$, and $v_2 \to v_4$ occur, and we prove that the arc $v_1 \to v_4$ must also occur. Note that while the vertices v_1, v_2, v_3, v_4 need not be distinct, we do have $v_1 \neq v_2$ and $v_3 \neq v_4$ since E has no multiple arcs. As before, we assume that the vertex v_λ in E corresponds to the pair (S_λ, T_λ) in the double partition. Hence, there exist vertices u, v, and w in D such that $u \in T_1 \cap S_3$, $v \in T_2 \cap S_3$, and $w \in T_2 \cap S_4$, and we must show that there exists a vertex $x \in T_1 \cap S_4$, thereby establishing the existence of the arc $v_1 \to v_4$ in E. Since S_3 and T_2 are not singleton sets and $v \in S_3 \cap T_2$, u, v and w are all associated with the same tri-level subgraph.

Whether they are sinks, sources or middle-level vertices depends upon which of the four sets S_1, S_2, T_3 and T_4 are empty. Because of the symmetry in the argument resulting from directional duality, only the following cases need to be considered:

(0) None of the sets is empty.

(i) Exactly one set is empty: (a) T_4, (b) T_3.

(ii) Exactly two sets are empty: (a) T_3 and T_4, (b) S_2 and T_4, (c) S_1 and T_4, (d) S_2 and T_3.

(iii) Exactly three sets sets are empty: (a) all except S_1, (b) all except S_2.

(iv) All four sets are empty.

We adopt the convention that if one of these four sets is non-empty, a vertex in it will be denoted s_1 in S_1, s_2 in S_2, t_3 in T_3, and t_4 in T_4.

We observe that if some of the vertices v_i are not distinct, then the proofs of the cases which follow also hold if these substitutions are made:

$$\begin{aligned}
&\text{if } v_1 = v_4, && \text{let } s_1 = w \text{ and } t_4 = u;\\
&\text{if } v_1 = v_3, && \text{let } s_1 = t_3 = u;\\
&\text{if } v_2 = v_3, && \text{let } s_2 = t_3 = v;\\
&\text{if } v_2 = v_4, && \text{let } s_2 = t_4 = w.
\end{aligned}$$

The diagrams in Fig. 10 are used to assist in the proofs; in some of the digraphs the notation $y(i, j)$ is used for a vertex; this means that y is a vertex in S_i and T_j.

Case (0). If none of the four sets S_1, S_2, T_3 and T_4 is empty, then there exist 2-diwalks in D from s_1 to t_3, from s_2 to t_3, and from s_2 to t_4. By (2), there is therefore a 2-diwalk $s_1 \to x \to t_4$, and the vertex x must be in both T_1 and S_4. (See Fig. 10.)

Case (i-a): $T_4 = \emptyset$. In this case w is not a sink neighbor of s_2; also $v_1 \neq v_2$ so that w is not a sink neighbor of s_1. Hence there is by our construction a corresponding sink neighbor x of s_1 in the same S_λ as w; that is, $x \in T_1 \cap S_4$.

Case (i-b): $T_3 = \emptyset$. In this case, u and v are sinks both in S_3. So, by our construction, they must have arcs to them from first-level vertices s_1 and s_2. Thus, there is a vertex c in the figure and so, by (2), there must exist a 2-diwalk from s_1 to t_4; its middle-level vertex must be in $S_4 \cap T_1$.

Case (ii-a): $T_3 = T_4 = \emptyset$. We have a vertex c as in the preceding case and, by condition (3) and the construction, a sink neighbor of s_1 must be paired with w.

Case (ii-b): $S_2 = T_4 = \emptyset$. Thus $w \in T_2 \cap S_4$ is an isolated vertex. Since s_1 is a first-level vertex associated with the same tri-level subgraph (and $w \in S_4$), there must be, by our construction, a sink neighbor of s_1 (that is, a vertex in T_1) in S_4.

Case (ii-c): $S_1 = T_4 = \emptyset$. In this case, w is a sink and u a source, and both are attached to the same tri-level subgraph. By condition (4) there is a corresponding

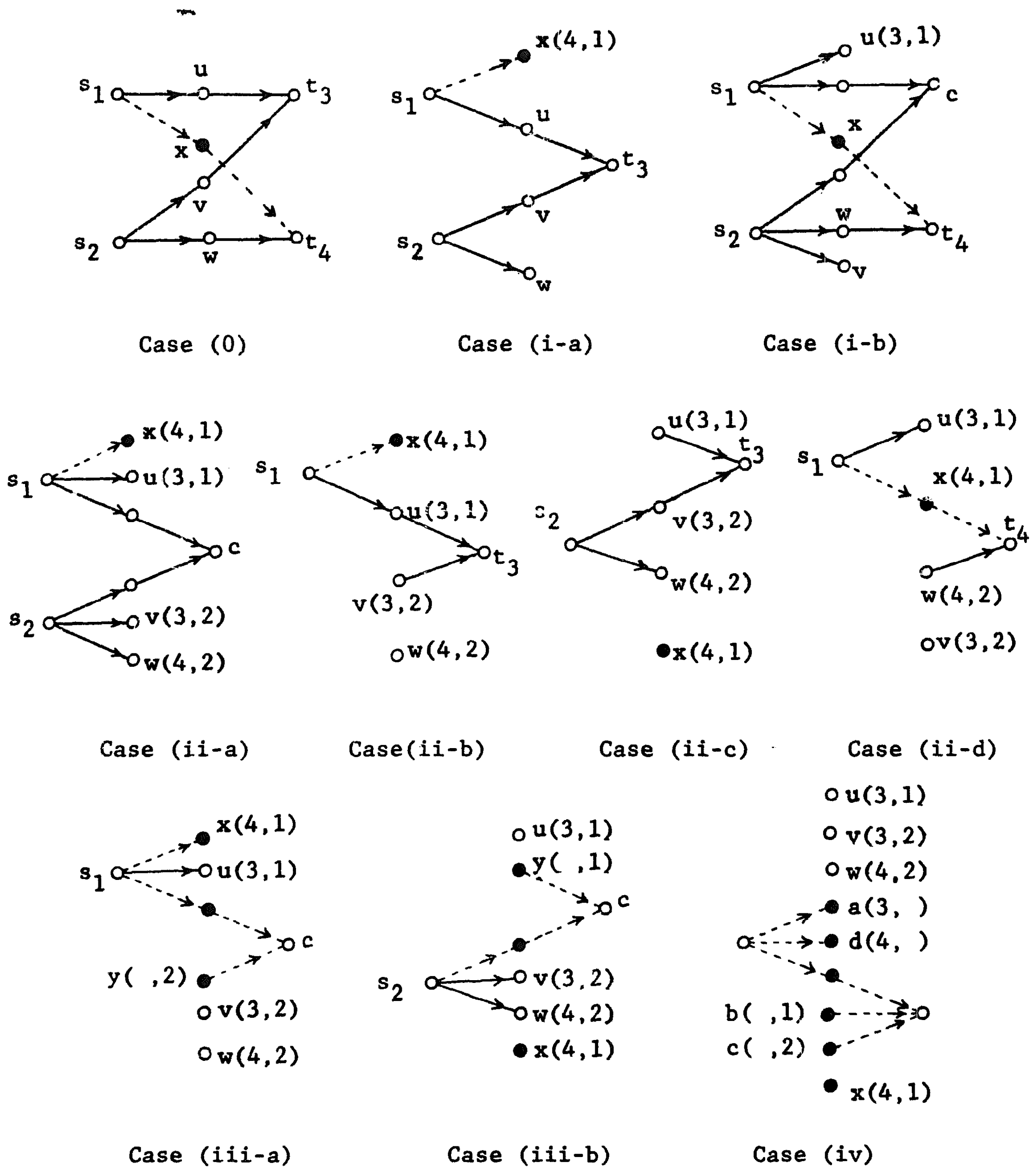


Fig. 10.

isolated vertex x, which by our construction is in $T_1 \cap S_4$.

Case (ii-d): $S_2 = T_3 = \emptyset$. In this case, v is an isolated vertex, u is a sink, and w a source, and they must, by our construction, be associated with a tri-level subgraph in which there is a 2-diwalk from s_1 to t_4. The middle-level vertex is in $S_4 \cap T_1$.

Case (iii-a): $S_2 = T_3 = T_4 = \emptyset$. Here u is a sink and v and w are isolated vertices. There must be a source neighbor of a last-level vertex c with $y \in T_2$ to account for $v \in S_3$ along with u. But then s_1 must have a sink neighbor x to account for w and y being in the same T_i. Hence $x \in S_4 \cap T_1$.

Case (iii-b): $S_1 = T_3 = T_4 = \emptyset$. In this case, u is an isolated vertex, and v and w

are sink neighbors of s_2. It follows as in the last case that there is a source $y \in T_1$ in conjunction with u and v, and hence there must be another isolated vertex x, for y and w. Therefore x is in the required two sets.

Case (iv): *All four sets are empty*. Here, all three vertices u, v, and w are isolated. Since u and v are both in S_3, there must also be a sink a in S_3 and two corresponding sources b and c in T_1 and T_2, all attached to the same tri-level subgraph. Similarly, and associated with the same subgraph, there must be a second sink neighbor at each first-level vertex, since v and w are both in T_2. It follows that there is a fourth isolated vertex x, and it must be in $S_4 \cap T_1$. This completes the proof that E is a line digraph, and thus it follows that D is a second-order line digraph.

3. Related results

In this second section we present results on characterizations of iterated line digraphs for two special classes of digraphs: (i) digraphs without loops or multiple arcs, and (ii) digraphs without sources or sinks. In the first case, the characterizations are just for first- and second-order line digraphs, but in the second case the characterizations are for all orders. Whenever possible, we try to give the characterization in terms of local structural conditions as, for instance, by putting in evidence, whenever possible, forbidden subgraphs.

Graph theorists are frequently interested in digraphs without loops or multiple arcs, such digraphs being called *simple*. In the next two theorems, we give necessary and sufficient conditions for a digraph to be the first- or second-order line digraph of some simple digraph. The first-order result was stated without proof by Beineke [1] and was also discovered by Zamfirescu [7], but because no proof appears in the literature (and because there have been several instances of results on line digraphs which were stated without proof turning out to be false), we give a proof here. As we observed earlier, a line digraph $D = L(E)$ never has multiple arcs (so a loop-free line digraph is actually a simple line digraph), and D has loops iff E has. Therefore, D must satisfy Heuchenne's condition while avoiding loops and multiple arcs.

Theorem 5. *A simple digraph D is the line digraph of some simple digraph iff the following conditions are satisfied.*

(1) *D has no subgraph isomorphic to any of the digraphs in Fig.* 11.

(2) *If D has a subgraph with the three solid arcs in Fig.* 12, *then the dashed arc must also be present.*

Proof. First, let D be the line digraph of some simple digraph E. Clearly, Heuchenne's condition (statement (3) of Theorem 1) implies that D must satisfy the condition (2). If D were to contain A_0 of Fig. 11 as a subgraph, there would

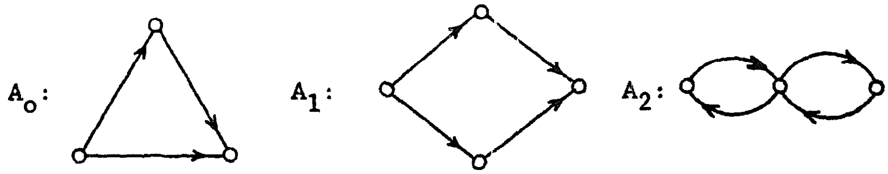


Fig. 11.

have to be a loop (at the top vertex) by Heuchenne's condition. Furthermore, if D were to contain A_1 or A_2, then any double partition of D would have some first set S_i and some second set T_j with two common elements corresponding to the two vertices with in- and out-valency both 1 in both A_1 and A_2, and consequently the connection digraph would have multiple arcs. Therefore, D can contain none of the three digraphs in Fig. 11, and condition (1) holds.

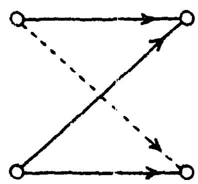


Fig. 12.

For the converse, we assume that D is a simple digraph satisfying (1) and (2). It is clear (see Fig. 3) that D satisfies Heuchenne's condition and hence is a line digraph. All that remains to be shown, therefore, is that if the connection digraph of every double partition of D has multiple arcs, then D must contain A_1 or A_2. But if that is the case, every double partition must have two pairs (S_i, T_i) and (S_j, T_j) with two vertices b and c in $T_i \cap S_j$ and with S_i and T_j non-empty, say a in S_i and d in T_j (e.g., if $T_j = \emptyset$, S_j could be split into singleton sets). Since D is loop-free, neither a nor d can equal b or c. Furthermore, if $a = d$, then D must contain a copy of A_2, while if $a \neq d$, it must contain A_1. This completes the proof.

The corresponding theorem for second-order line digraphs is of course considerably more complicated, but its proof follows similar lines.

Theorem 6. *A simple line digraph is the second-order line digraph of some simple digraph iff the following conditions are satisfied.*

(1) *D has no subgraph isomorphic to any of the digraphs in Fig.* 13.

(2) *If D has a subgraph with the solid arcs in any of the digraphs in Fig.* 14, *then the dashed* 2-*diwalk must also be present (these are the* 2-*diwalk versions of the digraphs in Fig.* 3).

(3) *For each subgraph of D of either of the first two types in Fig.* 15, *the vertices labeled a and b must have the same number of sink neighbors, and for each subgraph of either of the last two types, the vertices labeled c and d must have the same number of source neighbors.*

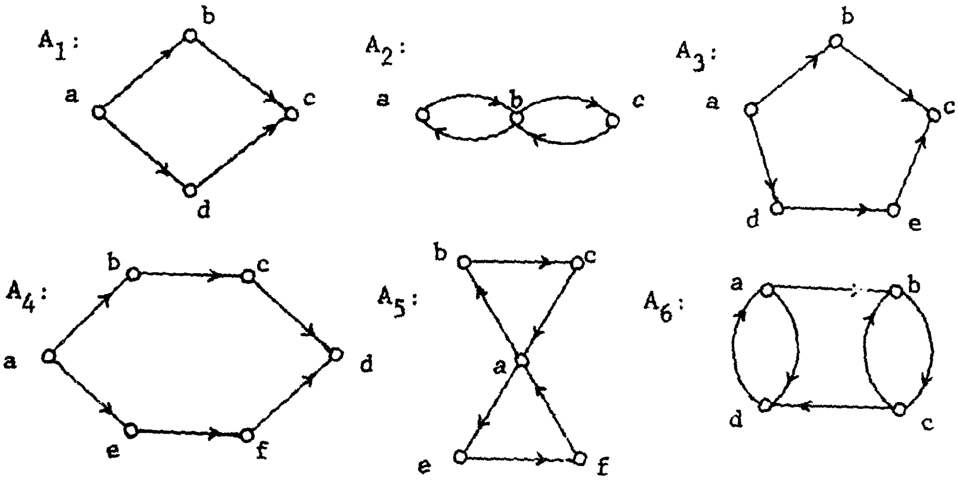


Fig. 13.

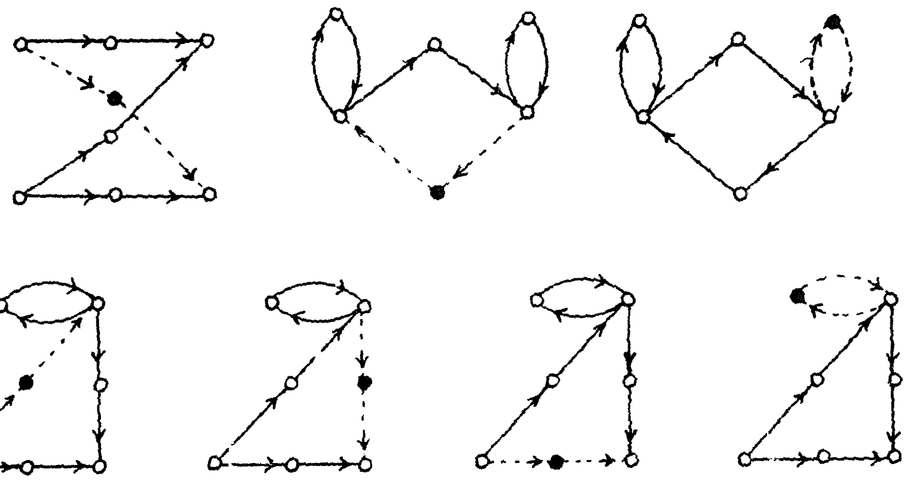


Fig. 14.

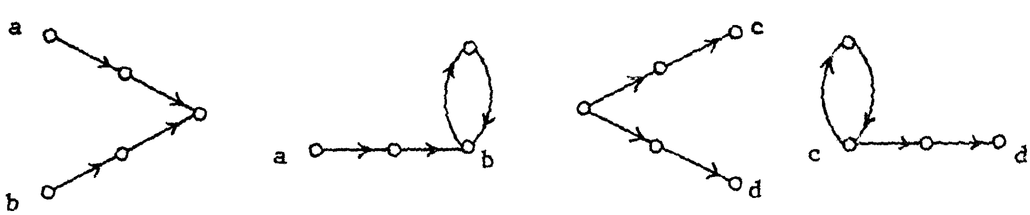


Fig. 15.

(4) *For each 2-diwalk $b \to d \to c$ of D of either type shown in Fig.* 16 ($d \neq b$, c), *there are associated pq isolated vertices, where p is the number of sink neighbors of b and q is the number of source neighbors of c and where non-$\sim$-related 2-diwalks have disjoint sets of associated isolated vertices.*

Proof. Assume first that D is the second-order line digraph of some simple digraph F. Conditions (2), (3), and (4) follow from the corresponding conditions of

Fig. 16.

Theorem 4. Furthermore, since D is the line digraph of $L(F)$ and $L(F)$ is simple, that D cannot contain A_1 nor A_2 follows from the preceding theorem. Suppose that D contains a subgraph isomorphic to A_3. Then there are 2-dipaths from d to c, from a to c and from a to e, so there must be one from d to e. This, however, implies the existence of a loop or a transitive triple, both forbidden, so D cannot contain A_3. Next, suppose D contains A_4 or A_5. In either case $L(F)$ would have loops, multiple arcs or a subgraph isomorphic to A_1 (in Fig. 11), and this cannot be. Further, if D contains A_6, then $L(F)$ contains loops, multiple arcs or a subgraph isomorphic to A_2, and this violates Theorem 5. This proves the necessity of (1)–(4).

Suppose now that D is a simple line digraph satisfying these four conditions. We first show that it is a second-order line digraph by establishing conditions (1) through (4) of Theorem 4. Clearly, (1) is satisfied. For (2) suppose that D has 2-diwalks $a \to u \to c$, $b \to v \to c$ and $b \to w \to d$. We need to show that there is also a 2-diwalk $a \to x \to d$. If $v = u$ or w, we can take $x = v$ since the first Heuchenne condition must be satisfied. So, suppose $v \neq u$ and $v \neq w$. Then also $u \neq w$; for otherwise D contains a subgraph isomorphic to A_1 (if $b = c$) or A_2 (if $b \neq c$). Similarly $a \neq b$ and $c \neq d$. Moreover, each of u, v and w must be different from all other vertices here. For $u \neq a$ or c, $v \neq b$ or c and $w \neq b$ or d since D has no loops. It follows that $u \neq b$, $w \neq c$ and $v \neq a$ or d since D does not contain transitive triples (see Fig. 3). Finally, since $a \neq b$, $c \neq d$ and D does not have a subgraph isomorphic to A_3 we have $u \neq d$ and $w \neq a$. Therefore, either all vertices are distinct, we have just one of the identifications $a = c$, $a = d$, $b = c$, and $b = d$, or we have either $a = c$ and $b = d$ or $a = d$ and $b = c$. These correspond to the seven graphs in Fig. 14 and in each case we have the existence of the desired vertex x and 2-diwalk $a \to x \to d$. Hence, D satisfies the second Heuchenne condition, (2) in Theorem 4.

Next, we observe that under our hypotheses, the only types of 2-diwalks which can occur in a tri-level subgraph are 2-dipaths or 2-dicircuits. Hence, because of the first and second Heuchenne conditions, (3) and (4) imply the corresponding conditions in Theorem 4. It follows that D is a simple second-order line digraph.

So let $D = L^2(F)$ where $E = L(F)$ is a simple digraph. Thus E must satisfy (2) and contain no subgraph isomorphic to A_0 of Fig. 11. Now we clearly can choose F so that all sources and sinks in E have valency one. But then, E contains no subgraph isomorphic to A_1; for if so there would exist an arc directed to the first vertex and an arc directed from the last vertex of A_1. Depending on whether these arcs are the same or different, D must contain either A_5 or A_4. Furthermore, if E were to contain A_2, then D would contain A_6. It thus follows by Theorem 5 that E is the line digraph of some simple digraph, of which D is thus the second-order line digraph. This completes the proof.

Another special class of digraphs which are frequently of interest are those which are just loop-free. As we observed earlier, the line digraphs of all such are simple. Necessary and sufficient conditions for simple digraphs to be first- or second-order line digraphs are similar to those given in Theorems 5 and 6. All one needs do is to omit some of the forbidden subgraphs. For the first-order case, the last two digraphs in Fig. 11 would not appear, while in the second-order case, the last three digraphs in Fig. 13 would not be present. The proofs are those of the corresponding theorems minus the portions involving the subgraphs omitted from Figs. 11 and 13.

We observe that in our main theorem, conditions (3) and (4) are needed only if the digraph has sources or sinks. Hence, Hemminger's original statement does provide necessary and sufficient conditions for a digraph to be a second-order line digraph if every vertex has positive in- and out-valencies. In fact, nth order line digraphs can be characterized in the same way. The proof of that fact relies heavily on two simple observations. The first is that if $\alpha : s \to a$ and $\beta : b \to t$ are arcs in a digraph E with line digraph D, then there is a k-diwalk from a to b in E iff there is a $(k+1)$-diwalk from α to β in D. The other is that if a digraph has no valency 0, then all vertices are on arbitrarily long diwalks.

Theorem 7. *Let D be a digraph in which no in- or out-valency is 0. Then D is an nth-order line digraph iff, for $k = 1, 2, \ldots, n$, the following conditions are satisfied.*

(1) *There are no multiple k-diwalks between any pairs of vertices.*

(2) *It satisfies the kth Heuchenne condition.*

Proof. We establish the sufficiency of these conditions using induction on n. The result holds for $n = 1$, and we assume it holds for $n = p$. Assume that D satisfies the hypotheses for $n = p+1$. Then D is a line digraph; and because there are no valencies equal to 0, it has a unique double partition $\{(S_\lambda, T_\lambda)\}_{\lambda=1}^{m}$. Let E be the corresponding connection digraph (so $D = L(E)$). It is sufficient to show that E has no multiple k-diwalks and satisfies the kth Heuchenne condition for all $k \leq p$. Suppose that for some such k, E does have two k-diwalks from a to b. Since a is not a source and b is not a sink, there must exist arcs $\alpha : s \to a$ and $\beta : b \to t$. Hence, D must have two $(k+1)$-diwalks from α to β, and since this contradicts our hypotheses, E can have no multiple k-diwalks for $k \leq p$. Similarly, if E has k-diwalks from a to c, from b to c, and from b to d, there are arcs $\alpha : r \to a$, $\beta : s \to b$, $\gamma : c \to t$, and $\delta : d \to u$. It follows that D has $(k+1)$-diwalks from α to γ, from β to γ, and from β to δ, and, by the $(k+1)$st Heuchenne condition, one from α to δ. Consequently, E must have a k-diwalk from a to d and hence must satisfy the kth Heuchenne condition for $k \leq p$. Therefore, by induction hypothesis, E is a pth-order line digraph, and D, as its line digraph, is of $(p+1)$st-order. This establishes the sufficiency.

The proof of the necessity is similar to that of (1) and (2) in Theorem 4 for general second-order line digraphs, and hence additional details will be omitted.

We leave as an open problem the characterization of nth-order line digraphs in general. From Theorem 4 it would appear that the statements of necessary and sufficient conditions could be quite complicated.

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References

[1] L.W. Beineke, On derived graphs and digraphs, in: H. Sachs et al., eds., Beitraege zur Graphentheorie (Teubner-Verlag, Leipzig, 1968) 17–23.

[2] F. Harary and R.Z. Norman, Some properties of line digraphs, Rend. Circ. Mat. Palermo (2) 9 (1960) 161–168; MR 24#A693.

[3] R.L. Hemminger, Line digraphs, in: Y. Alavi et al., eds., Graph Theory and Applications, Lectures Notes in Mathematics, Vol. 303 (Springer-Verlag, Berlin, Heidelberg, New York, 1972) 149–163; MR 51#243.

[4] R.L. Hemminger and L.W. Beineke, Line graphs and line digraphs, in: L.W. Beineke and R.J. Wilson, eds., Selected Topics in Graph Theory (Academic Press, London, New York) 271–305.

[5] C. Heuchenne, Sur une certaine correspondance entre graphes, Bull. Soc. Roy. Sci. Liège 33 (1964) 743–753; MR 30#5297.

[6] P.I. Richards, Precedence constraints and arrow diagrams, SIAM Rev. 9 (1967) 548–553.

[7] C. Zamfirescu, Lokale und globale Untersuchungen der Line-, Middle- und Total-Digraphen, Doctoral Dissertation, Technical University Aachen, Germany, 1977.